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NEGATIVE FREQUENCY

In its less abstract contexts, frequency is always regarded as a quantity devoid of sign; crudely, one might say that a sine-wave looks much the same whether the phase increases at ω rad/s or at $-\omega$ rad/s. One might alternatively say that as $\cos(-\omega t) = \cos \omega t$ and $\sin(-\omega t) = -\sin \omega t$, the distinction between positive and negative frequency is trivial, amounting at the most to a difference of phase. Or one might think of the vector representation of $\cos \omega t$ or $\sin \omega t$ as

$$\tfrac{1}{2}[\exp(+\mathrm{j}\omega t) \pm \exp(-\mathrm{j}\omega t)]$$

and say that both positive and negative frequencies exist together in any wave. Ignoring thus the sign of a frequency is undoubtedly justified in the vast majority of practical circumstances.

Practical problems: polyphase modulation. There are practical circumstances where some use can be made of what at first sight looks like a distinction between positive and negative frequencies. In a heterodyne receiver, for example, the intermediate-frequency signal has a frequency (ω_2) equal to the difference between the frequencies of the incoming signal and the local oscillator, ω_1 and ω_0 respectively. Thus $\omega_2 = \omega_1 - \omega_0$ say. But a signal in the "image channel" at frequency $\omega_0 - \omega_2$ also gives an i.f. of ω_2. Of course, $(\omega_0 - \omega_2) - \omega_0 = -\omega_2$, but normally no advantage can be taken of the idea that the i.f. is of "opposite sign" for the wanted signal and the image channel respectively. This conception is indeed meaningless in a normal circuit.

If the heterodyne process is carried out in a polyphase modulator, however, it is easily shown[1] that the sequence of the polyphase i.f. signal is opposite for the wanted and the image channels; and since sequence discriminators are easily made, the image channel can be rejected on this basis. It is tempting to think of this as discriminating between negative and positive frequency, and to confuse the direction of sequence with the sign of frequency; although there may be some practical justification for this, there is no theoretical justification.[2]

Theoretical problems: Circuits with time-varying parameters. In the analysis of linear circuits with time-varying parameters[3]—forming the basis of modulators and parametric amplifiers—there is considerable advantage in writing a modulation product frequency as $\omega_q + m\omega_p$, where ω_q is the signal frequency, ω_p is the local "carrier" or "pump" frequency, and m is an integer which may be positive or negative or zero. With this notation, the frequency obviously becomes negative in most likely circumstances whenever negative values of m are taken. The more obvious alternative notation, $n\omega_p \pm \omega_q$, where n is a positive integer or zero, has the small advantage that the frequency is positive in almost all likely circumstances, but has the very serious disadvantage that phase angles associated with the current or voltage at a particular frequency become reversed whenever a difference-frequency interaction product is formed. This leads to complication in solving the equations.[4]

Since in the analysis, a negative frequency $(\omega_q - n\omega_p)$ will be assumed to have a corresponding circuit impedance Z_{-n}, it is important to consider what is the relation of this impedance to the physical or measurable impedance Z_{n-} which the practical circuit has at the scalar (or positive) frequency $(n\omega_p - \omega_q)$. Alternatively, the latter may be specified in a particular problem, and we need to know what to insert in the equations.

It is known from Foster's Reactance Theorem[5] that reactance may always be expressed as an odd function of frequency, so that we may write $Z_{-n} = R_{-n} + \mathrm{j}X_{-n} = R_{-n} + \mathrm{j}\omega_{-n}\mathrm{f}(\omega^2{}_{-n})$ where ω_{-n} is a shortened notation for $(\omega_q - n\omega_p)$ and $f(\omega^2_{-n}) = f(\omega^2_{n-})$. Thus $Z_{-n} = R_{n-} - j\omega_{n-}f(\omega^2_{n-}) = Z_{n-}^*$ so that it is merely necessary to use the conjugate of the practical impedance in the theoretical analysis.

The band-pass/low-pass transformation. The idea that a low-pass filter of cut-off frequency ω_c behaves exactly as a band-pass filter of pass-band extending from a negative frequency $(-\omega_c)$ to a positive frequency ω_c is now widely used. It leads, for example, to the very useful result[6] that the response of an "ideal" filter (i.e. one with zero loss between cut-off frequencies ω_1 and ω_2 and infinite loss elsewhere, and with constant delay-time τ in the pass-band) to a suddenly-applied excitation at frequency $\frac{1}{2}(\omega_1+\omega_2)$—i.e. a unit step envelope—is

$$\left\{\frac{1}{2}+\frac{1}{\pi}\,\mathrm{Si}[\tfrac{1}{2}(\omega_2-\omega_1)(t-\tau)]\right\}\cos\tfrac{1}{2}(\omega_1+\omega_2)t$$

where Si is the sine-integral function. When ω_1 and ω_2 are both positive frequencies, the filter is band-pass. When $\omega_1 = -\omega_2 = -\omega_c$, the filter is low-pass, $\frac{1}{2}(\omega_1+\omega_2) = 0$, and the applied excitation is unidirectional (i.e. d.c.); the output is then

$$\frac{1}{2}+\frac{1}{\pi}\,\mathrm{Si}[\omega_c(t-\tau)].$$

To obtain this useful relationship, we have made use once more of the concept of negative frequency, and imagine a kind of image frequency-band below zero frequency. But again it is an abstraction; no measurements can be made in such a band.

Complex frequencies. It is well-known that the transfer operational function (T) of a two-terminal-pair network (in its most general form) may be written as the ratio of two polynomials in the operator p (according to the Heaviside operational calculus), e.g. $T = A(p)/B(p)$. (A similar expression in the variable s is used in the Laplace calculus.) The response of the network to any given excitation is determined by the values of the roots of $A(p) = 0$ and $B(p) = 0$, i.e. by the zeros and poles of the function T.

Now, in general, the roots of the polynomials are of the form $(\alpha+j\omega)$ or $(\alpha-j\omega)$. For example, in a circuit involving R, L and C in series, we obtain the roots of $B(p)$ as

$$p_1 = -\frac{R}{2L}+j\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}} \quad\text{and}\quad p_2 = -\frac{R}{2L}-j\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}.$$

Of course, in the network response each of these values of p occurs in the index of an exponential, and the exp $(+j\omega t)$ and exp $(-j\omega t)$ usually combine to give cos ωt or sin ωt, so that the exp (αt) represents the decay of a transient. But in advanced network theory it is found analytically convenient to regard the $(\alpha \pm j\omega)$ as corresponding to complex frequencies $(\pm\omega - j\alpha)$, and the location of these roots is plotted on what is called the "complex frequency plane". Thus, not only are negative as well as positive real frequencies used, but also imaginary frequencies. But this is an extreme type of mathematical abstraction, and the concept of a decay envelope as the waveform of an imaginary frequency is not likely to be of much value in visualizing circuit performance.

Conclusions. The concept of negative frequency (and even of complex frequency) has wide theoretical application. It must be accepted as an abstract concept, however, although in certain circumstances the distinction of sequence in a polyphase system may appear to correspond to a distinction of sign in the frequency.

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References

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