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Circuits with Time-Varying Parameters
(Modulators, Frequency-changers and Parametric Amplifiers)

By

Professor
D. G. TUCKER, D.Sc.
(*Member*) †

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Summary: It is shown that the theory of circuits with time-varying parameters may be presented in a manner sufficiently simple for undergraduate instruction, and that the basic conceptions of modulators with complex impedances and of parametric amplifiers may thus be made clear. Moreover, the treatment is general in nature, and so brings out the similarities and differences between circuits with time-varying resistance and those with time-varying inductance and capacitance. Non-linear effects are also touched upon.

1. Introduction

There is no doubt that, as with so many branches of engineering nowadays, the subject of circuits with time-varying parameters is often treated in such an advanced mathematical manner (see, e.g., the series of papers in the *I.R.E. Transactions on Circuit Theory*, March 1955) that it seems to be quite beyond the scope of undergraduate work. It is possible, however, to present the basic concepts very simply, and this simple treatment is adequate for the solution of many important practical circuits in the steady state. Examples of such circuits are amplitude modulators and parametric amplifiers.

There is nothing in Kirchhoff's Laws to prevent their application to time-varying circuits, and we therefore start with a statement of these laws for single loop and single node-pair circuits containing, in addition to ordinary impedances, elements of resistance, inductance or capacitance which are caused to vary with time.

2. Kirchhoff's Laws for Time-varying Circuits

The basic circuits are shown in Fig. 1. The single node-pair circuits can be re-arranged, as will soon become evident, as exact duals of the single loop circuits. Thus the equations may be written in dual pairs, where

$$Y = 1/Z, \qquad g(t) = 1/r(t).$$

Thus from Fig. 1(*a*), (i) and (ii) respectively,

$$E\cos\omega_q t = Z.i(t) + r(t).i(t) \qquad \text{......(1)}$$

$$I\cos\omega_q t = Y.v(t) + g(t).v(t) \qquad \text{......(2)}$$

From Fig. 1(*b*) (i) and Fig. 1(*c*) (ii) respectively,

$$E\cos\omega_q t = Z.i(t) + \frac{\mathrm{d}}{\mathrm{d}t}[L(t).i(t)] \qquad \text{......(3)}$$

$$I\cos\omega_q t = Y.v(t) + \frac{\mathrm{d}}{\mathrm{d}t}[C(t).v(t)] \qquad \text{......(4)}$$

† Electrical Engineering Department, University of Birmingham.

From Fig. 1(*c*) (i) and Fig. 1(*b*) (ii) respectively,

$$E\cos\omega_q t = Z.i(t) + \frac{1}{C(t)}\int i(t).\mathrm{d}t \qquad \text{......(5)}$$

$$I\cos\omega_q t = Y.v(t) + \frac{1}{L(t)}\int v(t).\mathrm{d}t \qquad \text{......(6)}$$

These equations are true for every instant of time.

The practical applications of these circuits are mainly to rectifier modulators for time-varying resistance, and to parametric amplifiers for time-varying inductance and capacitance.

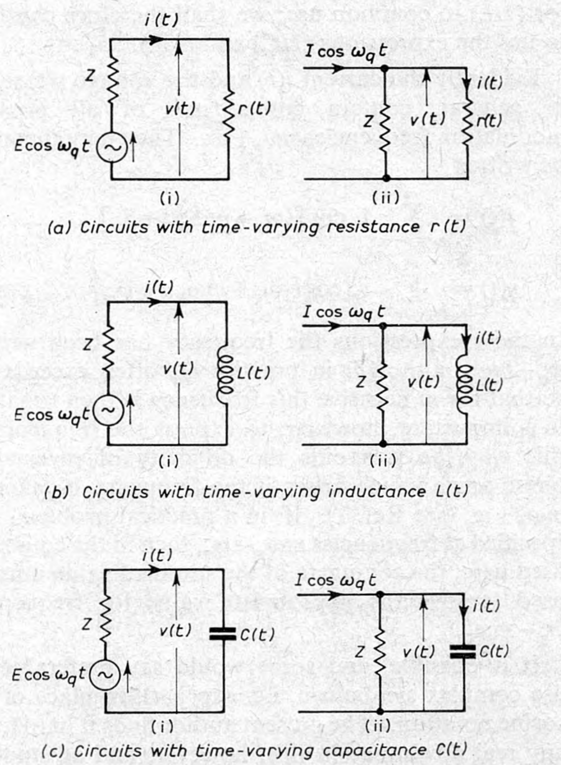

Fig. 1. Basic circuits.

Now, in practice, the time-varying element will vary in a periodic manner, and we call the fundamental angular frequency of this variation ω_p. Although it is not necessary, it is convenient in order to simplify the mathematical working (and also reasonably representative of practice) to restrict the Fourier series representing the time variation to cosine terms.

Thus

$$r(t) = \sum_{n=0}^{\infty} r_n \cos n\omega_p t \quad \ldots\ldots(7)$$

$$g(t) = \sum_{n=0}^{\infty} g_n \cos n\omega_p t \quad \ldots\ldots(8)$$

$$L(t) = \sum_{n=0}^{\infty} L_n \cos n\omega_p t \quad \ldots\ldots(9)$$

$$\frac{1}{L(t)} = \sum_{n=0}^{\infty} \left(\frac{1}{L}\right)_n \cos n\omega_p t \quad \ldots\ldots(10)$$

$$C(t) = \sum_{n=0}^{\infty} C_n \cos n\omega_p t \quad \ldots\ldots(11)$$

$$\frac{1}{C(t)} = \sum_{n=0}^{\infty} \left(\frac{1}{C}\right)_n \cos n\omega_p t \quad \ldots\ldots(12)$$

Note that, except for square-wave variations of $r(t)$, the g_n are not explicitly related to the r_n; similarly for $(1/L)$ and L and for $(1/C)$ and C. Although the term "elastance" with symbol S is often used for $(1/C)$, there seems to be no corresponding term and symbol for $(1/L)$ in common use; we shall therefore continue to use the expressions $(1/C)$ and $(1/L)$.

Evidently the current $i(t)$ and the voltage $v(t)$ must, in general, contain components of all possible modulation frequencies $n\omega_p \pm \omega_q$. They may therefore be written

$$i(t) = \sum_{m=-\infty}^{\infty} i_m \cos[(\omega_q + m\omega_p)t + \theta_m] \quad \ldots\ldots(13)$$

$$v(t) = \sum_{m=-\infty}^{\infty} v_m \cos[(\omega_q + m\omega_p)t + \phi_m] \quad \ldots\ldots(14)$$

In these expressions the frequency has been written $\omega_q + m\omega_p$, although in practice ω_p often exceeds ω_q, so that for m negative this frequency is then negative. It is important, however, to express the frequency in this way, as it avoids the difficulty of reversal of phase angle which arises if the frequency is taken as $m\omega_p \pm \omega_q$ (see Ref. 1). If, in a practical problem, Z is specified at frequencies $m\omega_p - \omega_q$, then in the equations used here, the conjugate of the specified value must be used to give the appropriate value for frequencies $\omega_q - m\omega_p$.

It is possible, and some would say preferable, to use complex symbolism, i.e. $\exp(j\omega t)$, in place of the cosine notation.[2] The present author finds it hard to see any real advantage in this, however, and as students probably think more easily in terms of cosine waves it seems preferable to use the cosine notation. Moreover, the complex symbolism using a single exponential for each frequency relies on the superposition theorem for its justification, and clearly applies only to linear circuits. It is surely very wrong, when it is avoidable, to teach students to think in a system which is restricted to linear circuits when so many real problems involve non-linearity. Often, too, the working of a problem is greatly complicated by the use of exponentials.

One other point requires noting before proceeding to the expansion of eqns. (1)–(6). The term $Z.i(t)$ is handled by using at each frequency $(\omega_q + m\omega_p)$ the value Z_m which Z has at that frequency, thus

$$Z.i(t) = \sum_{-\infty}^{\infty} Z_m i_m \cos[(\omega_q + m\omega_p)t + \theta_m] \quad \ldots\ldots(15)$$

and correspondingly for $Y.v(t)$.

3. Expansion for Time-varying Resistance

We shall deal with eqn. (1). Obviously eqn. (2) is handled in an exactly similar way.

The product $r(t).i(t)$ gives rise to a new series of terms, but without introducing any frequencies not already in $i(t)$. Thus

$$r_n \cos n\omega_p t . i_m \cos[(\omega_q + m\omega_p)t + \theta_m]$$
$$= \tfrac{1}{2} r_n i_m \cos\{[\omega_q + (m+n)\omega_p]t + \theta_m\} +$$
$$+ \tfrac{1}{2} r_n i_m \cos\{[\omega_q + (m-n)\omega_p]t + \theta_m\} \quad \ldots\ldots(16)$$

It is important to note that the phase angle still carries the same subscript as the current magnitude, i.e. although the frequency has been changed, the phase angle and magnitude still correspond. This means that it is quite unnecessary to retain the phase angles if the currents i_m are regarded as vector quantities.

Since the eqn. (1) is true at all instants of time, it must hold for each individual frequency taken separately. In the statement of equilibrium for each frequency, the cos $(\omega_q + m\omega_p)t$ will appear in every term and may be cancelled out. The expansion of eqn. (1) thus appears:

at frequency ω_q:

$$E = \ldots + \tfrac{1}{2} r_1 i_{-1} + (Z_0 + r_0) i_0 + \tfrac{1}{2} r_1 i_{+1} + \tfrac{1}{2} r_2 i_{+2} + \ldots \quad \ldots\ldots(17)$$

at frequency $\omega_q + \omega_p$:

$$0 = \ldots + \tfrac{1}{2} r_2 i_{-1} + \tfrac{1}{2} r_1 i_0 + (Z_{+1} + r_0) i_{+1} + \tfrac{1}{2} r_1 i_{+2} + \ldots \quad \ldots\ldots(18)$$

at frequency $\omega_q - \omega_p$:

$$0 = \ldots + (Z_{-1} + r_0) i_{-1} + \tfrac{1}{2} r_1 i_0 + \tfrac{1}{2} r_2 i_{+1} + \tfrac{1}{2} r_3 i_{+2} + \ldots \quad \ldots\ldots(19)$$

and at frequency $\omega_q + m\omega_p (m \neq 0)$:

$$0 = \ldots + \tfrac{1}{2} r_{|m|} i_0 + \tfrac{1}{2} r_{|m-1|} i_{+1} + \ldots$$
$$+ \tfrac{1}{2} r_1 i_{m-1} + (Z_m + r_0) i_m + \ldots \quad \ldots\ldots(20)$$

We thus have an infinite number of equations, each with an infinite number of terms. Explicit solution is not generally possible. But as soon as we can make some restrictions, solution becomes possible.

A very simple example of such restrictions is to specify that Z is infinite except at the two frequencies ω_q and $(\omega_q+\omega_p)$. This might, in practice, be a rectifier modulator of the "series" type with filters for source and load terminations, although it happens to be an inefficient arrangement and not recommended for general use. It is chosen here only for its analytical simplicity. Then all currents are zero except i_0 and i_{+1}. Equations (17) and (19) then become

$$E = (Z_0+r_0)i_0+\tfrac{1}{2}r_1 i_{+1} \quad ......(21)$$

$$0 = \tfrac{1}{2}r_1 i_0+(Z_{+1}+r_0)i_{+1} \quad ......(22)$$

giving the solution

$$i_{+1} = \frac{\tfrac{1}{2}Er_1}{\tfrac{1}{4}r_1^2-(Z_0+r_0)(Z_{+1}+r_0)} \quad ......(23)$$

It should be noted that although the two eqns. (17) and (19) are sufficient for solving for i_{+1} (and i_0), the other equations do not disappear, as some finite terms (including $Z_m i_m$) remain in each.

An example of a somewhat different kind is to specify that Z is a constant pure resistance (R) at all relevant frequencies, and that $r(t)$ is a square-wave. Then it can be shown by various methods[1,3] that no even-order products exist other than i_0—i.e. $i_m = 0$ when m is even and not zero. Equation (17) then becomes

$$E = (R+r_0)i_0+\tfrac{1}{2}\sum_{m=1,3,5,\ldots}^{\infty} r_m(i_{+m}+i_{-m}) \quad ......(24)$$

and eqn. (20) becomes, for m odd,

$$0 = (R+r_0)i_m+\tfrac{1}{2}r_m i_0 \quad ......(25)$$

These give immediately a solution for any particular current, e.g.

$$i_{+1} = \frac{-\tfrac{1}{2}Er_1}{(R+r_0)^2-\tfrac{1}{2}\sum_{m=1,3,5,\ldots}^{\infty} r_m^2} \quad ......(26)$$

Now as $r(t)$ is a square-wave,

$$\sum_{m=1,3,5,\ldots}^{\infty} r_m^2 = \frac{\pi^2}{8}r_1^2 \quad ......(27)$$

so that the solution is explicit.

It is clear that knowledge of the absence of even-order products is a great help in this problem, as it is indeed in many others.

4. Expansion for Time-varying Inductance or Capacitance

4.1. *Solution of Equations* (3) *and* (4)

Here we shall deal with eqn. (3). It is clear that the same working applies also to eqn. (4).

Now

$$\frac{d}{dt}[L(t).i(t)] = \frac{dL(t)}{dt}.i(t)+L(t).\frac{di(t)}{dt} \quad ......(28)$$

From eqn. (9),

$$\frac{dL(t)}{dt} = -\omega_p\sum_{n=1}^{\infty} nL_n \sin n\omega_p t \quad ......(29)$$

and from eqn. (11),

$$\frac{di(t)}{dt} = -\sum_{m=-\infty}^{\infty}(\omega_q+m\omega_p)i_m \sin[(\omega_q+m\omega_p)t+\theta_m] \quad ......(30)$$

Thus eqn. (3) becomes

$$E\cos\omega_q t = \sum_{m=-\infty}^{\infty} Z_m i_m \cos[(\omega_q+m\omega_p)t+\theta_m] - $$
$$-\omega_p\sum_{n=1}^{\infty} nL_n \sin n\omega_p t \times$$
$$\times \sum_{m=-\infty}^{\infty} i_m \cos[(\omega_q+m\omega_p)t+\theta_m] -$$
$$-\sum_{n=0}^{\infty} L_n \cos n\omega_p t \times$$
$$\times \sum_{m=-\infty}^{\infty}(\omega_q+m\omega_p)i_m \sin[(\omega_q+m\omega_p)t+\theta_m] \quad ......(31)$$

It will be observed that in all resultant terms, θ_m is always associated with i_m as in Section 3, but there are now sine as well as cosine terms. This can be taken care of by the use of the "j" operator with the sine terms, and a set of equations, in which only the vector coefficients appear, can then be formed, on the lines of, but more complicated than, eqns. (17)–(20).

If eqn. (31) is expanded, and the terms at each separate frequency are grouped together, it is readily seen that the following system of equations is obtained:

at frequency ω_q:

$$E = \ldots+\tfrac{1}{2}j\omega_q L_1 i_{-1}+(Z_0+j\omega_q L_0)i_0+\tfrac{1}{2}j\omega_q L_1 i_{+1}+\tfrac{1}{2}j\omega_q L_2 i_{+2}+\ldots \quad ......(32)$$

at frequency $\omega_q+\omega_p$:

$$0 = \ldots+\tfrac{1}{2}j(\omega_q+\omega_p)L_2 i_{-1}+\tfrac{1}{2}j(\omega_q+\omega_p)L_1 i_0+[Z_{+1}+j(\omega_q+\omega_p)L_0]i_{+1}+\tfrac{1}{2}j(\omega_q+\omega_p)L_1 i_{+2}+\ldots \quad ......(33)$$

and at frequency $\omega_q+m\omega_p$ ($m \neq 0$):

$$0 = \ldots+\tfrac{1}{2}j(\omega_q+m\omega_p)L_{|m|} i_0+\tfrac{1}{2}j(\omega_q+m\omega_p)L_{|m-1|} i_{+1}+\ldots+\tfrac{1}{2}j(\omega_q+m\omega_p)L_1 i_{m-1}+[Z_m+j(\omega_q+m\omega_p)L_0]i_m+ \quad ......(34)$$

As with the time-varying resistance, no explicit solution is possible for the general case. Particular cases in which specified restrictions are imposed are, however, soluble. A most important such case, which forms the basis of the parametric amplifier, is that in which Z, in Fig. 1(*b*) (i), is infinite at all frequencies except ω_q and $(\omega_q+\omega_p)$—(or $(\omega_q-\omega_p)$ as will be discussed later). Then i_0 and i_{+1} are the only currents which are not zero. Using eqns. (32) and (33) we obtain

$$E = (Z_0+j\omega_q L_0)i_0+\tfrac{1}{2}j\omega_q L_1 i_{+1} \qquad \text{......(35)}$$

and

$$0 = \tfrac{1}{2}j(\omega_q+\omega_p)L_1 i_0+[Z_{+1}+j(\omega_q+\omega_p)L_0]i_{+1} \qquad \text{......(36)}$$

whence

$$i_{+1} = \frac{-\tfrac{1}{2}j(\omega_q+\omega_p)L_1 E}{\tfrac{1}{4}\omega_q(\omega_q+\omega_p)L_1^2+(Z_0+j\omega_q L_0)[Z_{+1}+j(\omega_q+\omega_p)L_0]} \qquad \text{......(37)}$$

4.2. *Solution of Equations* (5) *and* (6)

Here, in order to retain the working for a time-varying inductance, we shall consider eqn. (6). Clearly eqn. (5) is worked out exactly similarly. It is also very important to work as often as possible in terms of admittance rather than impedance, as students seem to find this difficult, and eqn. (6) gives practice in this.

Using the expressions given in eqns. (10) and (14), eqn. (6) may be written

$$I\cos\omega_q t = \sum_{m=-\infty}^{\infty} Y_m v_m \cos[(\omega_q+m\omega_p)t+\phi_m] + \sum_{n=0}^{\infty}\left(\frac{1}{L}\right)_n \cos n\omega_p t \times \times \sum_{m=-\infty}^{\infty}\frac{v_m}{\omega_q+m\omega_p}\sin[(\omega_q+m\omega_p)t+\phi_m] \qquad \text{......(38)}$$

Expanding this into separate equations for each frequency, and regarding the coefficients v_m as vectors, we obtain:

at frequency ω_q:

$$I = \ldots+\frac{1}{2}\cdot\frac{1}{j(\omega_q-\omega_p)}\left(\frac{1}{L}\right)_1 v_{-1}+\left[Y_0+\frac{1}{j\omega_q}\left(\frac{1}{L}\right)_0\right]v_0+\frac{1}{2}\cdot\frac{1}{j(\omega_q+\omega_p)}\left(\frac{1}{L}\right)_1 v_{+1}+\ldots \qquad \text{......(39)}$$

and at frequency $\omega_q+m\omega_p$ $(m\neq 0)$:

$$0 = \ldots+\frac{1}{2}\cdot\frac{1}{j\omega_q}\left(\frac{1}{L}\right)_{|m|}v_0+\frac{1}{2}\cdot\frac{1}{j(\omega_q+\omega_p)}\left(\frac{1}{L}\right)_{|m-1|}v_{+1}+\ldots+\frac{1}{2}\cdot\frac{1}{j[\omega_q+(m-1)\omega_p]}\left(\frac{1}{L}\right)_1 v_{m-1}+ + \left[Y_m+\frac{1}{j(\omega_q+m\omega_p)}\left(\frac{1}{L}\right)_0\right]v_m+\ldots \qquad \text{......(40)}$$

It can be seen at once that this is quite different from eqns. (32)–(34); in particular, *different* frequencies occur in the coefficients of the various terms of one equation, instead of just the one frequency. Nevertheless, this set of equations is used in just the same way as before.

A particular case, which can also be used as the basis of a parametric amplifier, is that in which Y is infinite at all frequencies except ω_q and $(\omega_q+\omega_p)$, so that only v_0 and v_{+1} are finite, and only two equations are required to solve for v_{+1}.

This case gives

$$I = \left[Y_0+\frac{1}{j\omega_q}\left(\frac{1}{L}\right)_0\right]v_0+\frac{1}{2}\cdot\frac{1}{j(\omega_q+\omega_p)}\left(\frac{1}{L}\right)_1 v_{+1} \qquad \text{......(41)}$$

and

$$0 = \frac{1}{2}\cdot\frac{1}{j\omega_q}\left(\frac{1}{L}\right)_1 v_0+\left[Y_{+1}+\frac{1}{j(\omega_q+\omega_p)}\left(\frac{1}{L}\right)_0\right]v_{+1} \qquad \text{......(42)}$$

whence

$$v_{+1} = \frac{-\frac{1}{2}\cdot\frac{1}{j\omega_q}\left(\frac{1}{L}\right)_1 I}{\frac{1}{4}\cdot\frac{1}{\omega_q(\omega_q+\omega_p)}\left(\frac{1}{L}\right)_1^2+\left[Y_0+\frac{1}{j\omega_q}\left(\frac{1}{L}\right)_0\right]\left[Y_{+1}+\frac{1}{j(\omega_q+\omega_p)}\left(\frac{1}{L}\right)_0\right]} \qquad \text{......(43)}$$

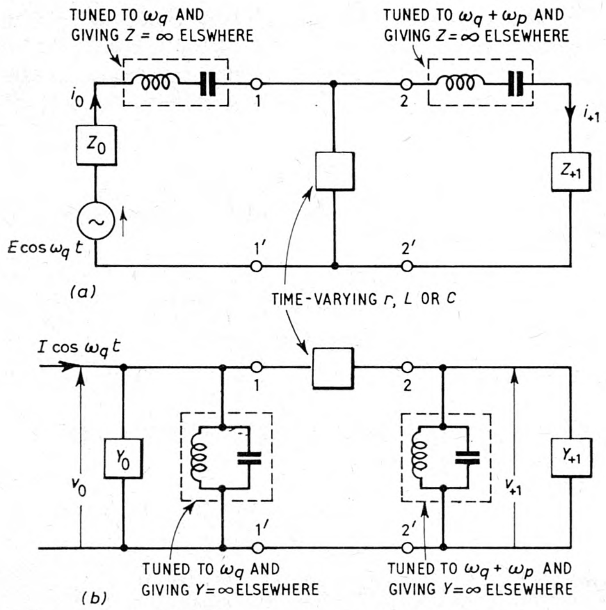

Fig. 2. Time-varying element in a circuit with
(*a*) only two non-zero currents
(*b*) only two non-zero voltages.

5. Further Consideration of Circuits Restricted to Two Non-zero Currents or Voltages

The examples of specific solutions which we have taken above have restricted the circuit to have either (a) non-zero currents at only two frequencies, or (b) non-zero voltages at only two frequencies. It is thus convenient to consider the circuit in its practical form as having separate parts for the input and output signals respectively, as shown in Fig. 2(*a*) and (*b*). It is clear that from the point of view of the previous analysis, since the two parts function at different frequencies, they may be superposed to give the single loops or single node-pairs of Fig. 1. Then Z_0 and Z_{+1} are merely the values of Z at frequencies ω_q and $(\omega_q+\omega_p)$; and similarly for Y. The tuned circuits shown are symbolic only, to indicate a practical way of obtaining an approximation to the required conditions.

We shall now show how the matching conditions and conversion loss or gain may be determined, and draw some general conclusions regarding these circuits.

5.1. *Time-varying Resistance*

Let us assume first of all that $Z_0 = R_0+jX_0$ is given, and that we require to find the value of $Z_{+1} = R_{+1}+jX_{+1}$ needed to give a conjugate match (i.e. maximum power transfer) at the terminals 2,2′ in Fig. 2(*a*). The circuit to the left of 2,2′ may be represented as a generator of e.m.f. E_{+1} and internal impedance $Z_{i,+1}$ at the frequency $\omega_q+\omega_p$. Obviously then, if we find $Z_{i,+1}$, we have to make $Z_{+1} = Z^*_{i,+1}$.

Now eqn. (21) gives us

$$i_0 = \frac{E-\frac{1}{2}r_1 i_{+1}}{Z_0+r_0} \qquad \ldots\ldots(44)$$

and since $E_{+1}-Z_{i,+1}i_{+1} = Z_{+1}i_{+1}$, then, from eqn. (22),

$$\begin{aligned} E_{+1}-Z_{i,+1}i_{+1} &= -\tfrac{1}{2}r_1 i_0 - r_0 i_{+1} \\ &= \frac{-\frac{1}{2}r_1 E}{Z_0+r_0} + \left[\frac{\frac{1}{4}r_1^2}{Z_0+r_0} - r_0\right] i_{+1} \end{aligned} \qquad \ldots\ldots(45)$$

so that

$$E_{+1} = \frac{-\frac{1}{2}r_1 E}{Z_0+r_0} \qquad \ldots\ldots(46)$$

and

$$Z_{i,+1} = r_0 - \frac{\frac{1}{4}r_1^2}{Z_0+r_0} \qquad \ldots\ldots(47)$$

$$= r_0 - \frac{\frac{1}{4}r_1^2(R_0+r_0)}{(R_0+r_0)^2+X_0^2} + j\,.\,\frac{\frac{1}{4}r_1^2 X_0}{(R_0+r_0)^2+X_0^2} \qquad \ldots\ldots(48)$$

Therefore for a conjugate match at 2,2′,

$$R_{+1} = r_0 - \frac{\frac{1}{4}r_1^2(R_0+r_0)}{(R_0+r_0)^2+X_0^2} \qquad \ldots\ldots(49)$$

and

$$X_{+1} = -\frac{\frac{1}{4}r_1^2 X_0}{(R_0+r_0)^2+X_0^2} \qquad \ldots\ldots(50)$$

When this match is obtained, the power in the output load

$$= |E_{+1}|^2/8R_{+1}$$

$$= \frac{\frac{1}{4}r_1^2E^2}{(R_0+r_0)^2+X_0^2} \Bigg/ \frac{8\{r_0[(R_0+r_0)^2+X_0^2]-\frac{1}{4}r_1^2(R_0+r_0)\}}{(R_0+r_0)^2+X_0^2} \qquad \ldots\ldots(51)$$

and for the given Z_0 and $r(t)$, the optimum conversion power-loss ratio

$$= \frac{\text{power available from signal source at } \omega_q}{\text{power in load at } \omega_q+\omega_p}$$

$$= \frac{r_0[(R_0+r_0)^2+X_0^2]-\frac{1}{4}r_1^2(R_0+r_0)}{\frac{1}{4}r_1^2 R_0} \qquad \ldots\ldots(52)$$

If the load Z_{+1} were given, and we had to find the value of Z_0 required for maximum power transfer, then we would calculate the input impedance at frequency ω_q looking from the left into terminals 1,1′ as

$$Z_{i0} = \frac{E-Z_0 i_0}{i_0}$$

$$= r_0 - \frac{\frac{1}{4}r_1^2}{Z_{+1}+r_0} \qquad \ldots\ldots(53)$$

We would then choose Z_0 to be the conjugate of Z_{i0}.

Equations (47) and (53) make it clear that the circuit is symmetrical, so that if Z_0 and Z_{+1} are both adjustable, the conversion loss will clearly be a minimum when $X_0 = X_{+1} = 0$ and

$$R_0 = R_{+1} = \sqrt{r_0^2 - \tfrac{1}{4}r_1^2}$$

The minimum conversion power-loss ratio is then

$$[r_0 + \sqrt{r_0^2 - \tfrac{1}{4}r_1^2}]^2 / \tfrac{1}{4}r_1^2 \qquad \text{......(54)}$$

It happens that the value of optimum circuit resistance thus obtained is usually inconveniently high; if $r(t)$ is a square-wave variation between values r_f (e.g. forward resistance of a rectifier) and r_b (e.g. backward resistance of a rectifier), then $R_0 = R_{+1} \simeq 0{\cdot}39\, r_b$ if it is assumed that $r_b \gg r_f$. The conversion loss may then be as low as 8·9 dB, but if the circuit resistance has to be reduced to some small fraction of r_b for practical reasons, the loss becomes high. If the time-varying resistance were, in fact, the opening and closing of a perfect switch, then it could be shown that the loss becomes infinite.

The method can, of course, be applied (but with more complexity) to other frequency-changer arrangements where more than two currents or voltages exist, and where lower losses may be obtained. For a tabulation of such arrangements and their minimum losses, see reference 1.

The minimum conversion power-loss ratio it is possible to have in any circuit with time-varying resistance is unity—i.e. no loss at all—and this occurs in a single-loop circuit when the following conditions are met:

(a) $r(t)$ is a square-wave function switching between values of zero and infinity,

(b) $Z = R_0$ at frequency ω_q, R_{+1} at frequency $(\omega_q + \omega_p)$, zero at all other odd-order product frequencies, and infinity at all other even-order product frequencies,

(c) $R_0 = (4/\pi^2)R_{+1}$.

There is, of course, a corresponding dual circuit.

This zero-loss condition is hardly a practical one, as the impedance requirements are almost impracticable; nor is it easily deducible from the general theory given here. But it is of basic theoretical importance in showing that it is, in principle, possible to convert all available signal power to another frequency; and it can be readily realized in practice with a ring modulator—i.e. with a 3-loop circuit. The matter is fully discussed by Belevitch.[4]

5.2. *Time-varying Inductance or Capacitance*

5.2.1. Use as a frequency-converter

The same method is used as that discussed in the previous section.

Working with the circuit of Fig. 2(*a*), and using time-varying inductance, we have, from eqns. (35) and (36), the following values for the effective generator at frequency $(\omega_q + \omega_p)$ seen to the left of terminals 2,2′:

$$E_{+1} = \frac{-\tfrac{1}{2}\mathrm{j}(\omega_q + \omega_p)L_1 E}{Z_0 + \mathrm{j}\omega_q L_0} \qquad \text{......(55)}$$

$$Z_{i,+1} = \mathrm{j}(\omega_q + \omega_p)L_0 + \frac{\tfrac{1}{4}\omega_q(\omega_q + \omega_p)L_1^2}{Z_0 + \mathrm{j}\omega_q L_0} \qquad \text{......(56)}$$

For a given Z_0, therefore, the value of

$$Z_{+1} = R_{+1} + \mathrm{j}X_{+1}$$

to give a conjugate match at 2,2′ (and hence maximum power transfer) is $Z^*_{i,+1}$ so that

$$R_{+1} = \frac{\tfrac{1}{4}\omega_q(\omega_q + \omega_p)L_1^2 R_0}{R_0^2 + (X_0 + \omega_q L_0)^2} \qquad \text{......(57)}$$

and

$$X_{+1} = -(\omega_a + \omega_v)\left\{L_0 - \frac{\tfrac{1}{4}\omega_q L_1^2(X_0 + \omega_q L_0)}{R_0^2 + (X_0 + \omega_q L_0)^2}\right\} \qquad \text{......(58)}$$

With this match, the power in the load

$$= |E_{+1}|^2 / 8R_{+1}$$

$$= \frac{\omega_q + \omega_p}{\omega_q} \, \frac{E^2}{8R_0} \qquad \text{......(59)}$$

so that the power *gain*

$$= \frac{\omega_q + \omega_p}{\omega_q} \qquad \text{......(60)}$$

Evidently, therefore, if $\omega_p \gg \omega_q$, this gain is very large. This is the basis of the parametric amplifier of the so-called upper-sideband "up-converter" type. It is an amplifier so long as the wanted output signal is of higher frequency than the input signal. The power gain is derived, of course, from the work done in varying L—commonly called the "pumping" action.

The gain given by eqn. (60) is obtained on making Z_{+1} a matched value for any given Z_0. It is interesting that no optimization of Z_0 and Z_{+1} in relation to $L(t)$ is required. In other words, if Z_0 is given, then a choice of Z_{+1} to give a conjugate match according to eqns. (57) and (58) will automatically give the maximum possible power gain of $(\omega_q + \omega_p)/\omega_q$. This is quite different from the behaviour of the circuit with time-varying resistance discussed earlier, since in that circuit the minimum conversion loss was obtained only when Z_0 and Z_{+1} were both chosen to have a particular, unique pair of values in relation to the Fourier coefficients of $r(t)$; mere conjugate matching with an arbitrary choice of Z_0 or Z_{+1} gave, in general, greater loss.

For two extreme cases, interesting results are obtained:

(a) $X_0 = -\omega_q L_0$ and $X_{+1} = -(\omega_q+\omega_p)L_0$

(This is the condition usually assumed in the literature.)

Then $$R_0 R_{+1} = \tfrac{1}{4}\omega_q(\omega_q+\omega_p)L_1^2 \qquad \ldots\ldots(61)$$

is the condition for maximum gain.

(b) $X_0 = 0$. Then for maximum gain,

$$\frac{R_{+1}}{R_0} = \frac{\omega_q+\omega_p}{\omega_q}\cdot\frac{\frac{1}{4}\omega_q^2L_1^2}{R_0^2+\omega_q^2L_0^2} \qquad \ldots\ldots(62)$$

and $$\frac{X_{+1}}{\omega_q L_0} = \frac{\omega_q+\omega_p}{\omega_q}\left[1-\frac{\frac{1}{4}\omega_q^2L_1^2}{R_0^2+\omega_q^2L_0^2}\right] \qquad \ldots\ldots(63)$$

Therefore $$\frac{R_{+1}}{R_0}+\frac{X_{+1}}{\omega_q L_0} = \frac{\omega_q+\omega_p}{\omega_q} \qquad \ldots\ldots(64)$$

Taking case (a), as it represents a usual practical arrangement, we can examine the effect of changing the signal frequency by a small amount. Since in this case $X_0+\omega_q L_0 \simeq 0$ for small changes in ω_q, we can write eqn. (58) as

$$X_{+1} \simeq -(\omega_q+\omega_p)\left\{L_0-\frac{1}{4R_0^2}\omega_q L_1^2(X_0+\omega_q L_0)\right\} \qquad \ldots\ldots(65)$$

Since X_0 is due to a capacitance (or, at any rate, its circuit is dominantly a capacitance), a small increase in ω_q will cause the expression between the large brackets to diminish in magnitude, so that the value of X_{+1} required for conjugate matching becomes less negative. But since X_{+1} is also necessarily due to a capacitance, this is just the way the value of X_{+1} would tend to alter due to the change of frequency. Thus an approximation to conjugate matching is obtained over a relatively wide frequency band. This means, therefore, that the up-converter is inherently a wide-band device.

Manley and Rowe[5] have shown that some general power relations exist in circuits with time-varying capacitance, and one of their results is that in this particular circuit the ratio of power absorbed in the load at frequency $(\omega_q+\omega_p)$ is always $(\omega_q+\omega_p)/\omega_q$ times the power *absorbed* from the signal source at frequency ω_q, irrespective of matching. The power gain, however, relates the load power to the *available* signal power, not to the signal power actually absorbed, and so to obtain the gain given by eqn. (60), matching is required.

If the same process (but using dual parameters) is applied to the circuit of Fig. 2(*b*) with time-varying inductance, working with eqns. (41) and (42), and putting $Y_0 = G_0+\mathrm{j}B_0$, then we find that for a conjugate match at 2,2′, we require $Y_{+1} = G_{+1}+\mathrm{j}B_{+1}$ where

$$G_{+1} = \frac{\frac{1}{4}\left(\frac{1}{L}\right)_1^2 G_0}{\omega_q(\omega_q+\omega_p)\left\{G_0^2+\left[B_0+\frac{1}{\omega_q}\left(\frac{1}{L}\right)_0\right]^2\right\}} \qquad \ldots\ldots(66)$$

and

$$B_{+1} = \frac{1}{\omega_q+\omega_p}\left\{\left(\frac{1}{L}\right)_0-\frac{\frac{1}{4}\left(\frac{1}{L}\right)_1^2\left[B_0+\frac{1}{\omega_q}\left(\frac{1}{L}\right)_0\right]}{\omega_q\left\{G_0^2+\left[B_0+\frac{1}{\omega_q}\left(\frac{1}{L}\right)_0\right]^2\right\}}\right\} \qquad \ldots\ldots(67)$$

This match gives a power gain of $(\omega_q+\omega_p)/\omega_q$ as for the previous circuit.

It will be found, indeed, that all the four circuits of Fig. 1(*b*) and (*c*), give the same performance, and expressions for matching, when proper allowance is made for the dual relationships. This means that an up-converter (and indeed, the negative-resistance parametric amplifier discussed below) can be made with time-varying inductance *or* capacitance, and with the circuit impedance made infinite *or* zero at the unwanted frequencies.

It is extremely interesting that the maximum power gain, as given by eqn. (60), is independent of the amount of inductance variation, as represented by L_1 or $(1/L)_1$. One would at first have supposed that as the variation was reduced, so also would the power delivered into the output circuit be reduced. But, taking the circuit of Fig. 2(*a*), the optimum resistance termination (R_{+1}) is seen from eqn. (57) to be a function of L_1, so that as L_1 is reduced, so also is R_{+1}. Thus, as we approach the limit, variations in the inductance (although small) are opposed (according to Lenz's law) by increased forces due to the almost short-circuited condition; this makes it possible for the power absorbed in producing the variations to remain constant in spite of the reduced variation. There is, of course, a discontinuity at the limit where L_1 actually becomes zero, since no gain can then be produced.

It can be shown that if the inductance is associated with a resistance representing its losses—as in practice it must be—the gain reduces continuously to zero as L_1 is reduced.

5.2.2. Negative-resistance parametric amplifier

It is evident that if the output frequency were $\omega_q-\omega_p$ instead of $\omega_q+\omega_p$, there would be less gain, and there could indeed be a power loss instead of a gain. But this case can be exploited, nevertheless, to give another kind of parametric amplifier. Suppose then that the system has Z infinite except at frequencies ω_q and $\omega_q-\omega_p$, according to Fig. 2(*a*), with time-varying inductance, and that we calculate the input impedance (Z_{i0}) seen from the signal source terminals 1,1′. Equations (35) and (36), adapted for frequency $\omega_q - \omega_p$, give

$$Z_{i0} = \frac{E - i_0 Z_0}{i_0} = j\omega_q L_0 + \frac{\frac{1}{4}\omega_q(\omega_q - \omega_p)L_1^2}{R_{-1} + j[X_{-1} + (\omega_q - \omega_p)L_0]}$$

$$= \frac{\frac{1}{4}\omega_q(\omega_q - \omega_p)L_1^2 R_{-1}}{R_{-1}^2 + [X_{-1} + (\omega_q - \omega_p)L_0]^2} +$$

$$+ j\left\{\omega_q L_0 - \frac{\frac{1}{4}\omega_q(\omega_q - \omega_p)L_1^2[X_{-1} + (\omega_q - \omega_p)L_0]}{R_{-1}^2 + [X_{-1} + (\omega_q - \omega_p)L_0]^2}\right\} \quad \ldots\ldots(68)$$

It is thus immediately clear that if $\omega_p \gg \omega_q$, then the real part of this input impedance is a *negative* resistance:

$$R_{i0} = -\frac{\frac{1}{4}\omega_q(\omega_p - \omega_q)L_1^2 R_{-1}}{R_{-1}^2 + [X_{-1} - (\omega_p - \omega_q)L_0]^2} \quad \ldots\ldots(69)$$

If then a load circuit operating at frequency ω_q is connected across terminals 1,1′, its resistive component may be arranged to be very nearly cancelled out by the negative resistance R_{i0}. A large output may then be maintained by applying only a small power from the signal source, and so a power amplifier has been obtained. The power gain can clearly be made as high as desired; if it is made infinite by making R_{i0} completely cancel out the load resistance, then clearly a self-oscillator is obtained.

It is evident that the negative resistance amplifier, by removing most of the resistance component of any tuned circuit in the signal path, is inherently a narrow-bandwidth device; in this respect it contrasts markedly with the up-converter, which is inherently a wide-band device, as previously explained.

Exactly corresponding results are obtained with the circuit of Fig. 2(*b*), and with a time-varying capacitance. For high-frequency use, the capacitance. amplifier is the more suitable in practice.[6,7]

An account of the history of parametric amplifiers, with a very large bibliography, is given by Mumford.[8]

6. More Complicated Circuit Configurations

In the case of the time-varying resistance, which represents the important practical circuits known as rectifier modulators, circuits comprising more than one loop or one node-pair are common[9,10]—e.g. the ring (or lattice) modulator. However, with due care, all the usual circuit theorems and transformations can be applied to them,[11] and in some circumstances very simple equivalent circuits can be found. For instance, with the assumption of a local carrier (or switching) oscillation having "half-cycle" symmetry—i.e. containing no even-order harmonics—the ring modulator can be shown to be equivalent to a single-loop or single-node-pair modulator.[1] Conditions for modulators to have an input impedance which is not time-varying can be specified[10] on the analogy of Zobel's constant-resistance networks. It is always important to remember, however, when making circuit transformations, that products such as $Z.r(t)$, which frequently arise, are not straight products; since Z can be regarded as a function of (d/dt), it is an operator, operating on $r(t)$. This frequently makes it difficult to make any use of the transformation.

7. Non-linear Effects in Circuits with Time-varying Parameters

The treatment so far given of time-varying parameters has merely assumed that the variation $r(t)$, $L(t)$ or $C(t)$ has been produced by some unspecified external agency. It is necessary, therefore, to give students some idea of how the variation is produced in practice, this being generally by the use of large-amplitude local signals which produce a time-varying bias in a non-linear element of r, L or C. The question will then inevitably be asked as to whether the information-signal voltages or currents, which also appear across or in the non-linear element, have any effect on the time-variation of the element, and whether they are subject to non-linear distortion due to the non-linearity of the element. The answer must be given that both effects do occur, and give rise to non-linear distortion.

A general treatment of this matter is prohibitively complicated (even if possible) for undergraduate courses. But with some special simplifying assumptions, an insight into the nature of the effects may be given in terms which are acceptable to undergraduates who have had a course in communication systems, and some previous contact with non-linearity.[12] One group of assumptions is as follows:

(a) the circuits are purely-resistive with time-varying resistance

(b) signal voltage or current small compared with bias wave

(c) nominally square-wave variation of $r(t)$ produced either

 (i) by cosine bias wave applied to bilinear rectifiers (i.e. rectifiers with a constant forward resistance and a constant back resistance, switching at zero applied voltage).

 or (ii) by square-wave bias applied to any kind of non-linear resistance.

Assumption (c) (i) permits the development of the idea of a square-wave time-variation, phase-modulated by the difference-frequency between signal and bias wave, so that either $r(t)$, or the corresponding time-varying transfer function $\phi(t)$, may be written approximately as

$$h_0+h_1\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\cos[(2n-1)\{\omega_p t-x_1\sin(\omega_p-\omega_{q1})t-x_2\sin(\omega_p-\omega_{q2})t-\ldots\}] \quad \ldots\ldots(70)$$

where ω_{q1}, ω_{q2}, etc. are various frequency-components of the information-signal of amplitudes x_1, x_2, etc. relative to the bias wave. The output signal consists basically of the product of this function and the input signal

$$e_1\cos\omega_{q1}t+e_2\cos\omega_{q2}t+\ldots \quad \ldots\ldots(71)$$

and it is clear that an infinite range of harmonics and intermodulation products is produced, with amplitudes dependent on both the relative signal amplitudes (x) and the order of modulation involved $(2n-1)$. A full account of this method is available elsewhere.[13]

Assumption (c) (ii) enables the lattice time-varying network (e.g. the ring modulator) to be regarded[13] as a non-linear but constant lattice followed by a reversing switch operating at frequency $\omega_p/2\pi$. The non-linear lattice can have a transfer function represented by a power-series, and the non-linear distortion products produced by this are easily calculated. They are then all multiplied by the switching function

$$\phi(t)=h_1\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\cos(2n-1)\omega_p t \quad \ldots\ldots(72)$$

to give the output spectrum.

It is clear these two processes (i) and (ii) give different kinds of result, and give a good indication of the complexity of non-linear analysis without introducing any processes beyond undergraduate level.

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